stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZW 149/80 DECEMBER
(DEPARTMENT OF PURE MATHEMATICS)

J. DE VRIES

LINEARIZATION, COMPACTIFICATION AND THE EXISTENCE OF NON-TRIVIAL COMPACT EXTENSORS FOR TOPOLOGICAL TRANSFORMATION GROUPS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics Subject Classification: Primary: 54H15
Secondary: 54C25, 54D35, 54C55

Linearization, compactification and the existence of non-trivial compact extensors for topological transformation groups*)

by

J. de Vries

ABSTRACT

In this paper we present a brief summary of some results concerning the topics, mentioned in the title.

KEY WORDS & PHRASES: *topological transformation group, equivariant mappings, equivariant embedding, linearization, compactification, injective object*

---

*) This report will be submitted for publication elsewhere.

1. Introduction. A *topological transformation group* (ttg), or a *G-space* is a triple $\langle G,X,\pi\rangle$, where G is a topological group, X is a topological space and π is a *continuous action* of G on X, i.e. $\pi: G \times X \to X$ is a continuous function such that $\pi(t,\pi(s,x)) = \pi(ts,x)$ and $\pi(e,x) = x$ ($s,t \in G$; $x \in X$; e denotes the identity of G). We shall use the following notation: $\pi^t x := \pi(t,x) =: \pi_x(t)$ for $t \in G$, $x \in X$. Then $\pi^t: X \to X$ in a homeomorphism, and $\pi_x: G \to X$ is continuous. If $\langle G,X,\pi\rangle$ and $\langle G,Y,\sigma\rangle$ are ttg's, then a mapping $f: X \to Y$ is called *equivariant* whenever $f \circ \pi^t = \sigma^t \circ f$ for all $t \in G$. A continuous equivariant mapping will also be called a *morphism of G-spaces*. For example, if $\langle G,X,\pi\rangle$ is a ttg such that G is locally compact, then the ttg $\langle G,C_c(G,X),\tilde{\rho}\rangle$ is defined by $\tilde{\rho}^t f(s) := f(st)$ ($s,t \in G$, $f \in C_c(G,X)$; local compactness of G guarantees the continuity of $\tilde{\rho}$), and the mapping $x \mapsto \pi_x: X \to C_c(G,X)$ is easily seen to be equivariant. For further definitions and elementary properties of ttg's and equivariant mappings, we refer to [17].

Let *Top* be the category of all topological spaces and let G be a fixed topological group. Then Top^G denotes the following category:

- objects: all ttg's $\langle G,X,\pi\rangle$;
- morphisms: all continuous equivariant mappings f between objects of Top^G.

Our general problem is as follows: consider *Top* as the category Top^G with $G = \{e\}$, and try to generalize results which are valid in *Top* to the case of Top^G with more general group G. In this paper the following results in *Top* will be considered:

- every metric space can be embedded in a Hilbert space;
- every Tychonov space can be embedded in a compact Hausdorff space;
- $I := [0,1]$ is an extensor for closed embeddings into normal spaces.

There is a certain overlap of this paper with [15]; the results have been obtained independantly of each other. In [15] also attention is paid to dimension theory, projective objects and absolutes. Our results in §2 and §3 have been published earlier, and proofs of the results of §4 will be published elsewhere.

2. Linearization. A ttg $\langle G,X,\pi\rangle$ will be called a *linear ttg* if X is a topological vector space and $\pi^t: X \to X$ is a linear homeomorphism (hence a topological isomorphism) for every $t \in G$. The following is easy to prove (cf. [17], 8.1.4; also [14]): *if* $\langle G,X,\pi\rangle$ *is any ttg, then it can equivariantly be embedded in a linear ttg* $\langle G,E,\sigma\rangle$ *with* E *a locally convex tvs if and only if* X *is a Tychonov space.* In fact, the linear ttg $\langle G,C_c(G,\mathbb{R}^k),\tilde{\rho}\rangle$ will do for any cardinal number $k \geq w(X)$, the weight of X. Now the following interesting problem arises: given a linear ttg $\langle G,E,\sigma\rangle$, determine the class $\mathcal{E}\langle G,E,\sigma\rangle$ of all ttg's $\langle G,X,\pi\rangle$ which can equivariantly be embedded in $\langle G,E,\sigma\rangle$. The following list contains some results in this direction

$\langle G,E,\sigma\rangle$	$\langle G,X,\pi\rangle \in \mathcal{E}\langle G,E,\sigma\rangle$ if	References
$\langle G,C_c(G),\tilde{\rho}\rangle$	G connected, X loc. cpt. sep. metr. and $X_G := \{x\in X: \pi^t x = x \text{ for all } t\in G\}$ is homeomorphic with a closed subset of $\mathbb{R}$.	[11]; also [12],[10], [6],[23]
$\langle \mathbb{R}, C_\nu^\infty, \tau\rangle$	$G = \mathbb{R}$, X sep. metr.	[21]
$\langle G, C_c(G \times G), r\rangle$	X Tychonov with $w(X) \leq L(G)$	[18]
$\langle G, L^2(G \times G, \nu \otimes \nu), r\rangle$	G σ-compact, X sep. metr.	[19],[21]
$\langle G, L^2(G,H,\nu), \tilde{\rho}\rangle$	G σ-compact, X metr., $w(X) \leq w(H)$	[16]
$\langle G,\mathbb{R}^\tau,\alpha\rangle$ $(\tau \geq w(G))$	G either compact, or G loc. cpt and $w(G) = \aleph_0$, X Tychonov and $w(X) \leq \tau$	[1] or [15]

Remarks. 1°. The first result in the above list is a special case of a more general theorem. It generalizes the classical BEBUTOV-KAKUTANI-HAJEK theorem.

2°. The space C_ν^∞ has been defined by D.H. CARLSON in [5].

3°. In the third and fourth results, $r^t f(u,v) := f(ut,vt)$ for $t,u,v \in G$ and $f: G \times G \to \mathbb{R}$ (one would obtain an isomorphic system if one would define

$r^t f(u,v) := f(ut,v)$; this has been done in [23;3.14]). For the measure ν in the 4th and 5th result, cf. [23;1.3, Example 7].

4^o. The 4th result follows from [21; last remark] under the additional condition $w(G) \leq \aleph_0$; one can get rid of this condition if one uses the inequality of Theorem 2 below.

5^o. Our 5th result generalizes and, in a certain sense, simplifies, earlier results of BAAYEN and DE GROOT [2]. It is the generalization to Top^G of the well known fact that every metric space can be embedded in a Hilbert space.

For more results, we refer to [23], where also for some results proofs are given which are shorter than those in the original references.

3. Compactification. A G-*compactification* of a G-space $\langle G,X,\pi\rangle$ is an equivariant continuous mapping $f:\langle G,X,\pi\rangle \to \langle G,Y,\sigma\rangle$ with dense range, where Y is a compact Hausdorff space. If f is an equivariant topological embedding, then it will be called a *proper* G-*compactification*. Necessary for the existence of a proper G-compactification is, that X is a Tychonov space and, still assuming G to be locally compact, this turns out to be also sufficient; cf. [20], [21] and also [15].

The original proof of the existence of proper G-compactifications in [21] is based on results in [18] concerning certain uniformities. A more elegant approach is the following one: in [20], Propositions 2.6 and 2.7, the following has been shown:

Theorem 1. *Let* $\langle G,X,\pi\rangle$ *be a ttg with* X *a Hausdorff space and* G *not necessarily locally compact. Then there is a natural* 1,1*-correspondence between* G-*compactifications of* $\langle G,X,\pi\rangle$ *and closed, invariant subalgebras of* C(X), *containing the constant functions, and which are contained in* $\pi UC(X)$. *A* G-*compactification is proper if and only if the corresponding subalgebra of* $\pi UC(X)$ *separates points and closed subsets of* X.

Here $\pi UC(X) := \{f \in C(X) : \{f \circ \pi_x\}_{x\in X}$ is equicontinuous on G$\}$, and "invariant" in the theorem is with respect to the set of all mappings $\tilde{\pi}^t$: $\tilde{\pi}^t: f \to f \circ \pi^t : C(X) \to C(X)$, $t \in G$. Now $\pi UC(X)$ itself turns out to be a closed invariant subalgebra of C(X), containing the constant functions. Consequently, to $\pi UC(X)$ corresponds the maximal G-compactification of $\langle G,X,\pi\rangle$ (i.e. its reflection into $Comp^G$; cf. [17;Section 4.3]. *Obviously, a proper* G-*compactification exists if and only if* $\pi UC(X)$ *separates points and closed subsets of* X. In the following situations this has been shown to be the case:

- G locally compact, and X a Tychonov space ([20] and [21]);

- G acts equicontinuously on X w.r.t. some separated uniformity for X; cf [13];
- a neighbourhood of e in G acts equicontinuously on X w.r.t. some separated uniformly for X (unpublished; see however [24]).

Clearly, the last result comprises the first and the second one. In the case of a locally compact group, slightly more can be shown (cf. [20]):

Theorem 2. *If* G *is locally compact and* X *is Tychonov, then* $\langle G,X,\pi\rangle$ *has a proper* G*-compactification* $f:\langle G,X,\pi\rangle \to \langle G,Y,\sigma\rangle$ *where* Y *is a compact Hausdorff space with*

$$w(Y) \le \max\{L(G/G_0),w(X)\}.$$

Here $G_0 := \{t \in G : \pi^t = \pi^e\}$ *and* $L(.)$ *denotes the Lindelöf degree.*

In the case that a neighbourhood of e acts equicontinuously on X, but G is not locally compact, the existence of a proper G-compactification of weight $\le \max\{w(G/G_0),w(X)\}$ can be proved; using an idea of H. LUDESCHER (personal communication), this can be improved to $w(Y) \le \max\{d(G/G_0),w(X)\}$ (observe that for locally compact groups, $L(G/G_0) \le d(G/G_0)$, so in that case the estimation in Theorem 2 is better).

Problem. Which nice properties of a ttg can be inherited by its compactifications? For example, if G acts equicontinuously on X, can $\langle G,X,\pi\rangle$ equivariantly be embedded in a ttg $\langle G,Y,\sigma\rangle$ with Y compact Hausdorff and σ an *equicontinuous* action of G on Y? The answer is yes, if orbit-closures in X are compact (use [7; Thm.7]).

4. Extensors. An *injective object* or an *extensor* for a morphism $\phi: A \to X$ in a category $\mathcal{C}$ is an object K in $\mathcal{C}$ such that for every morphism $f: A \to K$ there exists a (not necessarily unique) morphism $f': X \to K$ such that $f = f' \circ \phi$. It is well-known that in the category *Top* the unit interval I is an extensor for every closed embedding into a normal space. It is also known [10'] that *every metrizable compact convex subset of a locally convex tvs is an extensor in Top for every embedding for which* I *is an extensor*. For brevity, a metrizable, compact convex subset of a locally convex tvs will be called an *MC-set*. (Some of the results below using MC-sets can easily be modified to C-sets (= compact, convex subsets of locally convex tvs's) by restricting one's attention to closed embeddings into metrizable spaces, and using Dugundji's extension theorem.) We want to study (the existence of) non-trivial extensors for closed equi-

variant embeddings in the category $\mathcal{T}op^G$. For other results in this direction, cf. [1] and [22]. In particular, it follows from [22; Prop.4.1] that *for every MC-set* K *the ttg* $\langle G,C_c(G,K),\tilde{\rho}\rangle$ *is an extensor in* $\mathcal{T}op^G$ *for every closed equivariant embedding* ϕ: $\langle G,A,\pi\rangle \to \langle G,X,\pi\rangle$ *with* X *a normal space*. The disadvantage of this result is, that $C_c(G,K)$ has bad topological properties; in particular, it is not compact, and we would like to have a ttg with a *compact* Hausdorff phase space which is an extensor in $\mathcal{T}op$ at least for all closed equivariant embeddings ϕ:$\langle G,A,\pi\rangle \to \langle G,X,\pi\rangle$ with X compact Hausdorff. For *compact groups* G, the following is essentially due to GLEASON, and a similar proof can be given as in [13';1.4.3], using [4], §1.2,Cor. to Prop.5: *every ttg* $\langle G,K,\alpha\rangle$ *with* K *an* MC-*set is an extensor in* $\mathcal{T}op^G$ *for all equivariant closed embeddings* ϕ:$\langle G,A,\pi\rangle \to \langle G,X,\pi\rangle$ *with* X *normal*.

For general non-compact groups, I know of no satisfactory results. However, the following observations show, that dynamical properties of ttg's may play a role; we shall restrict ourselves to compact ttg's, although for non-compact ones something might be said as well, using results from §3 above.

If $\langle G,X,\pi\rangle$ is a ttg with X a compact Hausdorff space, then let $Q_X := \bigcap\{\overline{G\alpha} : \alpha \in \mathcal{U}\}$; here $\mathcal{U}$ is the uniformity for X, and $G\alpha := \{(\pi^t x,\pi^t y) : t \in G \ \& \ (x,y) \in \alpha\}$. Usually, Q_X is not an equivalence relation; let $Q_X^{\#}$ denote the smallest invariant closed subset of $X \times X$ which contains Q_X and which is an equivalence relation. More about the set $Q_X^{\#}$ (the so-called *regionally proximal relation* on X) and the space $X^{\#} := X/Q_X^{\#}$ (the so-called *maximal equicontinuous factor* of X) can be found in [3] or [8]. If A is a closed invariant subset of X, then A and also the embedding mapping $\phi : \langle G,A,\pi\rangle \to \langle G,X,\pi\rangle$ are called $Q^{\#}$-*admissible* whenever $Q_A^{\#} = Q_X^{\#} \cap (A \times A)$ or, equivalently, whenever the induced mapping $\phi^{\#} : A^{\#} \to X^{\#}$ is injective (the assignment $X \mapsto X^{\#}$ turns out to be functorial on $\mathcal{C}omp^G$). The following can be shown (details will be published elsewhere):

<u>Theorem 3</u>. *Let* $\langle G,X,\pi\rangle$ *be a ttg with* X *a compact Hausdorff space and* G *locally compact. Then the following conditions are equivalent for a closed invariant subset* A *of* X, $A \neq \phi$:

(i) A *is* $Q^{\#}$*-admissible;*

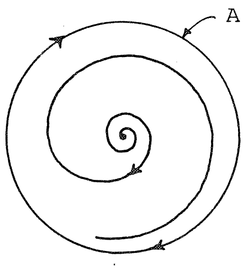
(ii) *Every equicontinuous ttg* $\langle G,K,\alpha\rangle$ *with* K *an MC-set is an extensor for the embedding* ϕ: $\langle G,A,\pi\rangle \to \langle G,X,\pi\rangle$.

The implication (i) $\Rightarrow$ (ii) *is also valid if* G *is not locally compact.*

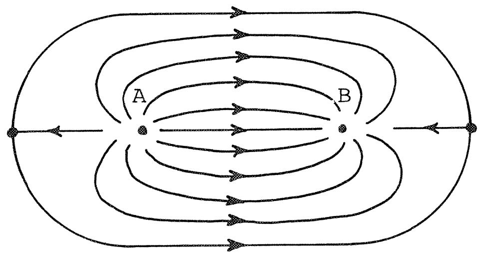
It follows from the theorem, that any equicontinuous ttg $\langle G,K,\alpha\rangle$ with K an MC-set is extensor in $\mathcal{T}op^G$ for every $Q^{\#}$-admissible closed invariant embedding into a ttg $\langle G,X,\pi\rangle$ with X compact Hausdorff. It can be shown that non-trivial equicontinuous ttg's $\langle G,K,\alpha\rangle$ with K an MC-set actually exist if and only if non-trivial equicontinuous ttg's $\langle G,X,\pi\rangle$ with X compact Hausdorff exist, if and only if the Bohr-compactification of G is non-trivial (G arbitrary).

<u>Problem</u>. To find characterizations for $Q^{\#}$-admissibility. The following one is a consequence of a result in [12'].. For any ttg, let $\pi\mathcal{A}(X) :=$ $\{f \in C(X) : \{f \circ \pi^t\}_{t\in G}$ is rel. cpt. in $C_u(X)\}$ (almost periodic functions on $\langle G,X,\pi\rangle$). Then for a closed invariant subset A of a ttg $\langle G,X,\pi\rangle$ with X compact Hausdorff the following are equivalent: (i) A is $Q^{\#}$ - admissible, and (ii) every $f \in \pi\mathcal{A}(X)$ can be extended to an $f' \in \pi\mathcal{A}(X)$.

In connection with this problem, observe that if A^{π} consists of one point (A a closed invariant subset of a compact Hausdorff G-space) then A is $Q^{\#}$-admissible. For conditions, guaranteeing that $A^{\#}$ is trivial, cf. [9]. The following pictures indicate examples where we have non-$Q^{\#}$-admissible closed subsets:



$A^{\#} = A$; $X^{\#} = \{\text{one point}\}$.



each of {A} and {B} is $Q^{\#}$-embedded;
$\{A\} \cup \{B\}$ is not $Q^{\#}$-embedded.

<u>References</u>.

[1] S.A. Antonian, Retracts in the category of G-spaces, Bull. Acad. Pol. Sci., Ser. Math., Astr., Phys, to appear

[2] P.C. Baayen & J. de Groot [1968], Linearization of locally compact transformation groups in Hilbert space, Math. Systems Theory <u>2</u> (1968),363-379.

[3] I.U. Bronstein [1979], Extensions of minimal transformation groups, Noordhoff, 1979.

[4] N. Bourbaki, Integration (Chap 6: integration vectorielle), Hermann, Paris, 1959.

[5] D.H. Carlson [1972], Universal dynamical systems, Math. Systems Theory <u>6</u> (1972),90-95.

[6] S.-S. Chen [1973], An extension of the Kakutani-Bebutov system, J. Differential equations 18(1975), 275-276.

[7] H. Chu, On semigroups in a non-compact transformation group, Math. Systems Theory 5(1971), 193-200.

[8] R. Ellis, Lectures on Topological Dynamics, W.A. Benjamin, Inc., New York, 1969.

[9] R. Ellis & H.B. Keynes, A characterization of the equicontinuous structure relation, Trans. Amer. Math. Soc. 161 (1971), 171-183.

[10] O. Hajek [1971], Representations of dynamical systems, Funkcial. Ekvac. 14(1971), 25-34.

[10'] G.J.O. Jameson, An elementary proof of the Arens and Borsuk extension theorems, J. London Math. Soc. (2), 14(1976), 364-368.

[11] A. Jaworski, The Kakutani-Bebutov theorem for groups, Diss. Univ. of Maryland, 1974.

[12] S. Kakutani [1968], A proof of Bebutov's theorem, J. Differential Equations 4(1968), 194-201.

[12'] M.B. Landstad [1972], On the Bohr compactification of a transformation group, Math. Z. 127 (1972), 167-178.

[13] H. Ludescher & J. de Vries, A sufficient condition for the existence of a G-compactification, Nederl. Akad. Wetensch. Proc. Ser. A., 83(1980), 263-268.

[13'] R.S. Palais, The classification of G-spaces, Memoirs of the Amer. Math. Soc., no.36, Providence, R.I., 1960.

[14] Yu. M. Smirnov [1976], On equivariant embeddings of G-spaces, Uspekhi Mat. Nauk. 31(1976), 137-147 (Russian); English translation in: Russian Math. Surveys 31(1976), 198-209.

[15] Yu. M. Smirnov, Compactifications, dimension and absolutes of topological transformation groups, in: Proc. of the conf. Topology and Measure III, held at Vitte (Hiddensee), DDR, oct. 1980.

[16] J. de Vries [1972], A note on topological linearization of locally compact transformation groups in Hilbert space, Math. Systems Theory 6(1972), 49-59.

[17] J. de Vries [1975a], Topological transformation groups 1, Mathematical Centre Tracts no.65, Mathematisch Centrum, Amsterdam, 1975.

[18] J. de Vries [1975b], Universal topological transformation groups, General Topology and Appl. 5(1975), 107-122.

[19] J. de Vries [1975c], A Universal topological transformation group in $L^2(G \times G)$, Math. Systems Theory 9 (1975), 46-50.

[20] J. de Vries [1977], Equivariant embeddings of G-spaces, in: J. Novak (ed.), General Topology and its Relations to Modern Analysis and Algebra IV, Part B (Proc. 4th Prague Topological Symposium, 1976), Prague 1977, pp.485-493.

[21] J. de Vries [1978], On the existence of G-compactifications, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978), 275-280.

[22] J. de Vries [1979], Topics in the theory of topological transformation groups, in: P.C. Baayen & J. van Mill (eds.), Topological structures II, MC Tract no. 115, Mathematisch Centrum, Amsterdam, 1979, pp.291-304.

[23] J. de Vries, Linearization of actions of locally compact groups, to appear in Proc. Steklov Inst. Math., vol.154.

[24] J. de Vries, Compactifications of topological transformation groups, Tagungsbericht 25/1980, Mathematisches Forschungsinstitut Oberwolfach.